Optimal Control in the Presence of State Uncertainty*

Milton Cunguara, Tomás Oliveira e Silva
IEETA, DETI - University of Aveiro
Campus Santiago, 3810-193
Aveiro, Portugal
{milton.cunguara,tos}@ua.pt

Paulo Pedreiras
IT, DETI - University of Aveiro
Campus Santiago, 3810-193
Aveiro, Portugal
pbrp@ua.pt

Abstract

The problem of optimal control given full state knowledge has been solved many years ago and is very well documented. Nonetheless, in most control settings it is virtually impossible to know the exact state value. To deal with this fact, two concepts were introduced: ***certainty equivalence*** *and* ***separation*** *principles. In this paper, it is shown that the longstanding certainty equivalence principle does hold in the presence of state uncertainty, leading in practice to sub-optimal control. Furthermore, it is also presented a novel mechanism that takes into account the effect of the sate estimation error in the control. Although its optimality cannot be established at the moment, it is proven that in the presence of state uncertainty the new mechanism outperforms classical approaches.*

1. Introduction

Even though some very significant steps have been made to find optimal control approaches, e.g. Bell's equation, many practical situations cannot be mapped to the conditions of applicability of such works. The set of such conditions includes, but is not limited to, state estimation errors, actuation jitter, whether the next actuation period will have a new actuation value, etc.

The standard approaches used in the presence of state estimation error are the *certainty equivalence* and *separation* principles. The *certainty equivalence principle* states that the optimal control value can be found by first finding the optimal control policy given that the state variable is known, and then substituting the state variable by its estimate in the control policy. The *separation principle* states that estimation and control are independent. Note that the *certainty equivalence principle* builds up on the *separation principle*, since it implies that control and estimation can be done separately. However, it is possible that the *separation principle* holds while the *certainty equivalence principle* does not, e.g. if the optimal control policy is a function of the estimation covariance. This latter observation is in the basis of the approach presented in this paper.

The *certainty equivalence principle* was believed to be applicable even to systems with state estimation error, and became the *de facto* standard in control practice. It shaped modern control theory being now completely widespread. However, the control policy attained from state estimations has fundamental differences with respect to the control policy attained when complete knowledge of the state is assumed and then substituted by its estimation, as will be shown in this paper. This difference stems from the fact that the estimation error may have a vector component that can be minimized by the controller.

The inadequate application of the certainty equivalence principle to systems with state estimation error can be, in the author's opinion, traced back to the fact that [15] showed that in systems in which the *certainty equivalence principle* holds the state error covariance matrix is not a function of the control value (i.e. no dual effect). However, the converse is not true, since it is possible to make corrections to the state in such a way that the state error covariance matrix becomes a function of the state, as will be shown in section 3.

Nevertheless, the novel approach presented in this paper is also not optimal since, by showing that the control performance can be improved by taking into account the estimation, it is also shown that different estimation covariance matrices lead to different qualities of control, thereby there is room for further improvement if the covariance matrix (i.e. the observation matrix) is chosen taking into account the control.

The reminder of this paper is organized as follows. Section 2 presents the related work. Section 3 introduces the novel mechanism for realizing optimal control in the presence of state estimation uncertainty and presents a theoretical proof that the proposed methodology outperforms the classical approaches. Finally, Section 4 presents the main conclusions and lines for future developments.

*This work was partially supported by the Portuguese Government through FCT - "Fundação para a Ciência e a Tecnologia" in the scope of project Serv-CPS -PTDC/EEA-AUT/122362/2010 and Ph.D. grant - SFRH/BD/62178/2009/J53651387T4E

2. Related Work

Optimal control is a discipline that is arguably around 90 years old and started with the work of Pontryagin and Richard Bellman, with Bellman's contribution being

$$V_k(x) = x_k' Q x_k + \min_{u_k} \left(u_k' R u_k + x_{k+1}' P_{k+1} x_{k+1} \right) \quad (1)$$

were $V_k = x_k' P_k x_k$ is the value function to be minimized, x is the state, u is the control signal, and Q and R are the wighting/cost matrices. This equation assumes that the system has no error whatsoever, thus being, from a control practice standpoint, essentially an interesting mathematical curiosity with limited usefulness. This changed when it got combined with Kalman filters, leading to the *Linear Quadratic Gaussian Regulator*. Decoupling estimation and control was justified by the certainty equivalence principle, which states that if $u_k = -L_k x_k$, i.e. the state is perfectly known, then if only a state estimation ($\hat{x}_{k|k}$) is available the optimal control value is $u_k = -L_k \hat{x}_{k|k}$ [16]. In [15] it was shown that the certainty equivalence principle holds if and only if the estimation error covariance matrix is not a function of the state i.e.

$$\frac{\partial E\left[\left(\hat{x}_{k|k} - x_k\right)' \left(\hat{x}_{k|k} - x_k\right)\right]}{\partial x_k} = \mathbf{0} \quad (2)$$

Recent contributions in this line of work show that the certainty equivalence principle is still widely accepted (e.g. [1,3,5,8–13]), though some works put its applicability in question for certain system classes. E.g. [6] and [7] show that in systems in which the parameters are unknown or because the system is non-linear, the use of the certainty equivalence leads to suboptimal control sequences, hence it is proposed an *heuristic* called *partial* certainty equivalence. Similar remarks were made in [4] regarding self-tuned controllers.

[14] reaches conclusions similar to the ones presented in this paper, however not providing a controller gain expression nor an optimal correction for the estimation. This article only points out, with significant mathematical background, that the correction of the state estimate $\hat{x}_{k|k}$ causes the certainty equivalence principle to *not hold*. Moreover, the state estimation error model used in [14] (order 1 IIR filter) is not common in practical control settings. Similar effects are also pointed out in [2].

3. Optimal Control in the Presence of State Uncertainty

This section presents the novel mechanism for *optimal control* in the presence of state uncertainty. However, due to space limitations, only the intuition and a sketch of the mathematical formulation is presented. A more detailed and formal presentation of the methodology herein described is available in a technical report accessible at *http://www.ieeta.pt/ pedreiras/tmp/opt-control-state-uncert.pdf.*

The main difference between this work and the *classical* approaches is centered around the acceptance of the *certainty equivalence principle*. Hence, instead of accepting it as given, the optimal control policy was computed starting from the state estimates, as opposed to computing starting from the actual state variable and then substitute its respective estimate. If the results were equal, then the certainty equivalence principle should hold.

Consider the minimization of

$$J = x_N' Q_k x_N + \sum_{k=0}^{N-1} x_k' Q_k x_k + u_k' R_k u_k \quad (3)$$

were Q_k and R_k are the weighting/cost matrices of the states and input at time k, respectively. Also, normally, $R_N = 0$ and $Q_k, R_k \geq 0$ (semi-positive definite). According to Bell's equation

$$V_k(x) = x_k' Q_k x_k + \min_{u_k} \left(u_k' R_k u_k + x_{k+1}' P_{k+1} x_{k+1} \right) \quad (4)$$

in which $J = V_0(x)$, $V_k(x) = x_k' P_k x_k + c_k$, c_k is not a function of u_k and $P_N = Q_N$.

Consider also the dynamic equation:

$$x_{k+1} = A x_k + B u_k + w_k \quad (5)$$

The solution of equation (4) given equation (5) is the control law:

$$u_k = -(B' P_{k+1} B + R_k)^{-1} B' P_{k+1} A x_k \quad (6)$$

However, since x_k is normally not available, it is substituted by

$$u_k = -(B' P_{k+1} B + R_k)^{-1} B' P_{k+1} A \hat{x}_{k|k} \quad (7)$$

which is the *classical optimal control* law.

However, if it assumed at start that x_k will not be available, and that the control law will be (pseudo-)linear, i.e. $u_k = -L_k \hat{x}_{k|k}$, the dynamic equation becomes

$$x_{k+1} = A x_k - B L_k \hat{x}_{k|k} + w_k \quad (8)$$

or

$$x_{k+1} = (A - B L_k) x_k - B L_k e_{k|k} + w_k \quad (9)$$

with $e_{k|k} = \hat{x}_{k|k} - x_k$. Equation (9) makes it evident that L_k can be used to minimize the effects of state uncertainty in cost function.

Returning to equation (4), and substituting equation (9) and $u_k = -L_k \hat{x}_{k|k}$, then

$$V_k(x) = x_k' Q_k x_k + \min_{L_k} \Big[\left(L_k \hat{x}_{k|k} \right)' R_k L_k \hat{x}_{k|k} + \left(A x_k - B L_k \hat{x}_{k|k} + w_k \right)' P_{k+1} \left(A x_k - B L_k \hat{x}_{k|k} + w_k \right) \Big] \quad (10)$$

Note that the minimization is carried out in order to L_k. Furthermore, L_k is the matrix that minimizes $E\left[V_k(x)\right]$, as opposed to minimize the stochastic variable $V_k(x)$. Unless stated otherwise, in the remaining of this paper all minimizations refer to their respective expectations. For clarity of presentation, the expectation sign will not be explicitly presented.

The derivative of the expectation of equation (10) with respect to L_k is

$$\frac{\partial V_k(x)}{\partial L_k} = 2\hat{x}_{k|k}\hat{x}'_{k|k}L'_kRL_k + 2\hat{x}_{k|k}\left(Ax_k - BL_k\hat{x}_{k|k}\right)' P_{k+1}(-B) \quad (11)$$

The value of L_k that minimizes equation (10) also makes $\frac{\partial V_k(x)}{\partial L_k} = 0$. Let $V_k^*(x) = \min V_k(x)$ and $L_k^* = \arg V_k^*(x)$. Then, after rearranging the terms and transposing

$$(R_k + B'P_{k+1}B)L_k^*\hat{x}_{k|k}\hat{x}'_{k|k} = B'P_{k+1}Ax\hat{x}'_{k|k} \quad (12)$$

Equation (12) is an important stepping stone. First, if there is no state estimation error, i.e. $\hat{x}_{k|k} = x_k$, then $L_k = \mathsf{L}_\mathsf{k} x_k x'_k \left(x_k x_k\right)^\dagger$, where L_k is the controller gain of *classical optimal control* and $(\cdot)^\dagger$ represents the pseudo-inverse. This means that in this case the computed control value is equal to the classical optimal one, since $u_k = L_k x_k$ and $x_k x'_k \left(x_k x_k\right)^\dagger x_k = x_k$.

3.1. Computing matrix L_k

Equation (12) implies a perfect knowledge of x_k, thus it is unsuitable to compute matrix L_k. To overcome this hurdle, $x_k\hat{x}'_{k|k}$ is used instead, which can be done by noticing that

$$x_k\hat{x}'_{k|k} = \hat{x}_{k|k}\hat{x}'_{k|k} - \left(e_{k|k}e'_{k|k} + e_{k|k}x'_k\right) \quad (13)$$

and

$$e_{k|k} = (I_n - K_kC)\left(Ae_{k-1|k-1} + w_{k-1}\right) + K_kv_k \quad (14)$$

$$x_k = (A - BL_{k-1})x_{k-1} - BL_{k-1}e_{k-1|k-1} + w_{k-1} \quad (15)$$

then

$$S_k = (I_n - K_kC)\,AS_{k-1}\left(A - BL_{k-1}\right)' + (I_n - K_kC)\left(-A\Sigma_{k-1|k-1}\left(BL_{k-1}\right) + Q_w\right) \quad (16)$$

were $\Sigma_{k|k} = E\left[e_{k|k}e'_{k|k}\right]$ and $S_k = E\left[e_{k|k}x'_k\right]$. Note, however, that in the standard Kalman filter $\Sigma_{k|k} + S_k \xrightarrow{k\to\infty} \mathbf{0}$, therefor the gains of this approach would be achieved only in the beginning and as time moved forward it would start to behave as a conventional controller. To solve this problem a new estimator that does not have this property and that is cappable of providing estimation of quality similar/better than the Kalman filter is also under development. By manipulating the equations above, the following relation can be obtained:

$$L_k^*\hat{x}_{k|k} = \mathsf{L}_\mathsf{k}\left(I_n - \left(\Sigma_{k|k} + S_k\right)E\left[\hat{x}_{k|k}\hat{x}'_{k|k}\right]^{-1}\right)\hat{x}_{k|k} \quad (17)$$

which, in turn, can be understood as

$$L_k^*\hat{x}_{k|k} = \mathsf{L}_\mathsf{k}\tilde{x}_{k|k} \quad (18)$$

in which

$$\tilde{x}_{k|k} = \hat{x}_{k|k} - E\left[e_{k|k}\middle|\hat{x}_{k|k}\right] \quad (19)$$

or in its expanded form,

$$\tilde{x}_{k|k} = \left(I_n - \left(\Sigma_{k|k} + S_k\right)E\left[\hat{x}_{k|k}\hat{x}'_{k|k}\right]^{-1}\right)\hat{x}_{k|k} \quad (20)$$

which implies that this novel controller performs a correction of the state estimate. This correction is based on the fact that, due to the presence of feedback, both the state and its estimate are correlated with past state estimation errors, which in turn are correlated with the present state estimate error. Hence, it is possible to estimate past error, which can be used to estimate the present error, thus making a correction.

3.2. Computing $V_k^*(x)$ and P_k

In the last section a novel value for L_k^*, given P_{k+1}, was derived. This section intends to derive the remaining aspects of the optimal controller. To this end, first notice that

$$\frac{\partial V_k(x)}{\partial L_k}\Big|_{L_k = L_k^*} = 0 \quad (21)$$

Hence, it also follows that

$$\begin{aligned}&\frac{1}{2}trace\left(\frac{\partial V_k(x)}{\partial L_k}L_k\right)\Big|_{L_k = L_k^*} = 0\\ &= \left(L_k^*\hat{x}_{k|k}\right)' R_kL_k^*\hat{x}_{k|k} + \left(Ax_k - BL_k^*\hat{x}_{k|k}\right)'\\ &P_{k+1}\left(-BL_k^*\hat{x}_{k|k}\right)\end{aligned} \quad (22)$$

Thus, $V_k^*(x)$ can be written as

$$\begin{aligned}V_k^*(x) =& V_k^*(x) - \frac{1}{2}trace\left(\frac{\partial V_k(x)}{\partial L_k}L_k\right)\Big|_{L_k = L_k^*}\\ =& x'_kQ_kx_k + \left(Ax_k - BL_k^*\hat{x}_{k|k}\right)'P_{k+1}Ax_k + \\ & trace\left(Q_wP_{k+1}\right)\end{aligned} \quad (23)$$

where, $Q_w = E\left[w_kw'_k\right]$. The last term is due to the equality $E\left[z'Mz\right] = trace\left(E\left[zz'\right]M\right)$.

Notice that the second term of the last equation still has a term in $\hat{x}_{k|k}$, which in turn does not match the initial form of $V_k^*(x)$, i.e.

$$V_k^*(x) = x'_kP_kx_k + c_k \quad (24)$$

However, with a few additional manipulations, $V_k^*(x)$ can be written as in equation (24), in which

$$P_k = Q + (A - BL_k^*)'P_{k+1}A \quad (25)$$

and

$$c_k = trace\left(\left(Q_w - A(BL_k^*S_k)'\right)P_{k+1}\right). \quad (26)$$

Note that P_k referred to in equation (25) is different from P_k in *classical optimal control*, since they are computed from different control gain matrices.

3.3. Theoretical Comparison with *Classical Optimal Control*

Past subsections provided a derivation of a novel controller. This section proves that this novel controller never performs worst than the *Classical Optimal Control*. The conditions on which both controllers have the same expected performance are also provided.

Equation (23) presented the value of $V_k(x)$ given that the controller was designed with the approach presented in this paper. It is easy to show that $V_k(x)$ given that the *classical optimal control* is employed with an uncertain state is

$$V_k^*(x) = x_k' \mathsf{P_k} x_k + trace\left(\left(A\Sigma_{k|k}\left(B\mathsf{L_k}\right)' + Q_w\right)\mathsf{P_{k+1}}\right) \tag{27}$$

Note that it was written $\mathsf{P_k}$ which intends to distinguish it from the P_k that appears when the proposed controller is used. In the same notation, the value of $V_k(x)$ (given that the controller had the value presented in this paper, see) is

$$V_k^*(x) = x_k' \mathsf{P_k} x_k + trace\Big(\Big(-Ax_k\left(G_k\hat{x}_{k|k} - x_k\right)'(B\mathsf{L_k})' + Q_w\Big)\mathsf{P_{k+1}}\Big) \tag{28}$$

with G_k^* defined by $L_k^* = \mathsf{L_k} G_k^*$. Therefore,

$$V_k(x)\big|_{L_k=\mathsf{L_k}} - V_k(x)\big|_{L_k=L_k^*} = trace\Big(A\Big(\Sigma_{k|k} + x_k\left(G_k^*\hat{x}_{k|k} - x_k\right)'\Big)(B\mathsf{L_k})'\,\mathsf{P_{k+1}}\Big) \tag{29}$$

Equation (29) is obviously non-negative. Hence, the presented controller is never outperformed by the *classical optimal control*. Note also that $x_k'\left(G_k^*\hat{x}_{k|k} - x_k\right)$ is the minimum of $trace\left(\left(G_k\hat{x}_{k|k} - x_k\right)'\left(G_k\hat{x}_{k|k} - x_k\right)\right)$, whereas $trace\left(\Sigma_{k|k}\right)$ is equal to the same function when $G_k = I_n$.

4. Conclusions and Future Work

In this paper it was shown that in general linear systems the certainty equivalence principle does not hold. This is so because there is a transmission from the state estimation error to the state that causes a correlation between them, and in turn can be used to find a better state estimate (a correction). Such a correction makes the state estimate error covariance matrix a function of the state, thereby, making the certainty equivalence principle not hold.

A novel controller that takes this fact was derived. It was also proved that such novel controller is never outperformed by the *classical optimal controller*.

Future lines of research consist in carrying out the experimental assessment and validation of the novel method proposed in the paper, as well as in the optimization of algorithms. Also, the extension of this line of work for the estimation framework.

References

[1] L. Bao, M. Skoglund, and K. Johansson. A scheme for joint quantization, error protection and feedback control over noisy channels. In *American Control Conference, 2007. ACC '07*, pages 4905 –4910, july 2007.

[2] Y. Bar-Shalom. Stochastic dynamic programming: Caution and probing. *Automatic Control, IEEE Transactions on*, 26(5):1184 – 1195, oct 1981.

[3] S. Battilotti. Robust detectability from the measurements plus state feedback stabilization imply semiglobal stabilization from the measurements. *Automatic Control, IEEE Transactions on*, 51(9):1542 –1547, sept. 2006.

[4] C. Charalambous. Stochastic nonlinear minimax dynamic games with noisy measurements. *Automatic Control, IEEE Transactions on*, 48(2):261 – 266, feb. 2003.

[5] C. Ebenbauer, T. Raff, and F. Allgower. Certainty-equivalence feedback design with polynomial-type feedbacks which guarantee iss. *Automatic Control, IEEE Transactions on*, 52(4):716 –720, april 2007.

[6] N. Filatov and H. Unbehauen. Design of improved adaptive controllers using partial certainty equivalence principle. In *American Control Conference. Proceedings of the 1998*, volume 2, pages 1068 –1072, jun 1998.

[7] N. Filatov and H. Unbehausen. Adaptive predictive control policy for nonlinear stochastic systems. *Automatic Control, IEEE Transactions on*, 40(11):1943 –1949, nov 1995.

[8] L. P. Hansen and T. J. Sargent. Certainty equivalence and model uncertainty. *Proceedings, Board of Governors of the Federal Reserve System (U.S.)*, pages 17–38, 2005.

[9] T. Li and J.-F. Zhang. Decentralized adaptive games for large population coupled arx systems with unknown coupling strength. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 3110 –3115, dec 2010.

[10] Z. Ma, D. Callaway, and I. Hiskens. Decentralized charging control for large populations of plug-in electric vehicles: Application of the nash certainty equivalence principle. In *Control Applications (CCA), 2010 IEEE International Conference on*, pages 191 –195, sept. 2010.

[11] A. Molin and S. Hirche. On lqg joint optimal scheduling and control under communication constraints. In *Decision and Control, 2009. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 5832 –5838, dec. 2009.

[12] G. Nair, F. Fagnani, S. Zampieri, and R. Evans. Feedback control under data rate constraints: An overview. *Proceedings of the IEEE*, 95(1):108 –137, Jan 2007.

[13] C. Ramesh, H. Sandberg, L. Bao, and K. Johansson. On the dual effect in state-based scheduling of networked control systems. In *American Control Conference (ACC), 2011*, volume 29, pages 2216 –2221, july 2011.

[14] E. T. Swanson. Signal extraction and non-certainty-equivalence in optimal monetary policy rules. *Macroeconomic Dynamics*, 8:27 –50, Jan 2004.

[15] E. Tse and Y. Bar-Shalom. Generalized certainty equivalence and dual effect in stochastic control. *Automatic Control, IEEE Transactions on*, 20(6):817 – 819, Dec 1975.

[16] H. Witsenhausen. Separation of estimation and control for discrete time systems. *Proceedings of the IEEE*, 59(11):1557 –1566, nov 1971.